Journal of Geometry and Physics 35 (2000) 157–182

q-KP hierarchy, bispectrality and Calogero–Moser systems

Plamen Iliev[1]

Department of Mathematics, Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium

Received 19 August 1999

Abstract

We show that there is a one-to-one correspondence between the q-tau functions of a q-deformation of the KP hierarchy and the planes in Sato Grassmannian Gr. Using this correspondence, we define a subspace Gr_q^{ad} of Gr, which is a q-deformation of Wilson's adelic Grassmannian Gr^{ad}. From each plane $W \in Gr_q^{\mathrm{ad}}$ we construct a bispectral commutative algebra $\mathcal{A}_W^q$ of q-difference operators, which extends to the case $q \neq 1$ all rank one solutions to the bispectral problem. The common eigenfunction $\Psi(x, z)$ for the operators from $\mathcal{A}_W^q$ is a q-wave (Baker–Akhiezer) function for a rational (in x) solution to the q-KP hierarchy. The poles of these solutions are governed by a certain q-deformation of the Calogero–Moser hierarchy. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 58F07; 35Q53; 39A70

Sub. Class: Quantum groups

Keywords: KP hierarchy; q-Deformations; Tau functions; q-pseudo-difference operators

1. Introduction

In [16], Frenkel proposed a q-deformation of the Nth KdV hierarchy which is Hamiltonian with respect to the quantum Poisson algebra $\mathcal{W}_q(\mathfrak{sl}_N)$ defined in [17]. A similar deformation of the KP hierarchy was obtained by Khesin et al. [30], who considered a certain q-deformation of the Lie algebra of pseudo-differential operators on the circle, see also [35].

In [24], a slightly different deformation of the KP hierarchy was proposed. It was shown that by making an appropriate shift in the arguments of the classical Schur polynomials, one obtains rational solutions of the deformed hierarchy. This result was extended in [1,25],

E-mail address: iliev@agel.ucl.ac.be (P. Iliev)

[1] Supported by a FSR Grant at the University of Louvain.

0393-0440/00/$ – see front matter © 2000 Elsevier Science B.V. All rights reserved.
PII: S0393-0440(00)00006-1

where it was proved that the same shift in any classical tau function leads to a solution of the deformed hierarchy.

In the present paper we complete the study of the q-tau functions, by showing in Theorem 2.1 that the shift mentioned above characterizes the q-tau functions in the ring of formal power series. Thus, we establish, in fact, a one-to-one correspondence between the q-tau functions and the planes in Sato Grassmannian.

As a first application of this result we construct a q-deformation of Wilson's adelic Grassmannian Gr_q^{ad}, which parametrizes rank one commutative bispectral algebras of q-difference operators. A q-difference operator $L(x, D_{q,x})$ is called *bispectral* if it has a family of eigenfunctions $\Psi(x, z)$ that is also a family of eigenfunctions of some q-difference operator $B(z, D_{q,z})$ in the spectral parameter z, i.e.

$$L(x, D_{q,x})\Psi(x, z) = f(z)\Psi(x, z), \tag{1.1}$$

$$B(z, D_{q,z})\Psi(x, z) = \theta(x)\Psi(x, z). \tag{1.2}$$

Here $D_{q,x}$ denotes the usual q-derivative operator acting on functions of x

$$D_{q,x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

In the limit $q \to 1$, L and B become ordinary differential operators. In this context the problem was posed and completely solved for L of order 2, in the pioneering work of Duistermaat and Grünbaum [15]. It turns out that this problem is intimately related with several actively developing areas of mathematics: integrable systems [15,41,42] and their master symmetries [44], the representation theory of Virasoro and $W_{1+\infty}$ algebras [8], Huygens principle [9], to mention only a few.

Our construction of Gr_q^{ad} is inspired by Wilson's approach [41] to the bispectral problem. In view of the works of Burchnall and Chaundy [11–13] and Krichever [31], one may consider any operator $L(x, \partial_x)$ as an element of a maximal commutative algebra $\mathcal{A}$ of differential operators. An important invariant of such an algebra is its *rank*, i.e., the greatest common divisor of the orders of the operators in the algebra. In [41] Wilson found a beautiful characterization of all rank one solutions to the bispectral problem. He proved that a maximal rank one commutative algebra $\mathcal{A}$ of ordinary differential operators is bispectral if and only if the curve Spec $\mathcal{A}$ is rational and unicursal (i.e. all singularities are cusps). The bispectrality is a consequence of an extra symmetry in Gr^{ad} called the *bispectral involution*. Roughly, this is the map which exchanges the role of the arguments in the Baker–Akhiezer function. In the framework of Sato Grassmannian, the rank one bispectral algebras are parametrized by an adelic Grassmannian Gr^{ad}, whose points correspond to solutions of the KP hierarchy, arising from unicursal rational curves by Krichever's construction. These solutions are nothing but the rational solutions of the KP hierarchy [32,40,42].

In the last few years, the original results of Duistermaat–Grünbaum and Wilson have been extended in several directions. Bakalov et al. [7] and Kasman and Rothstein [28] constructed bispectral algebras of ordinary differential operators of any rank (see also [6] for an abstract version of the bispectral problem and further examples). In a different vein, Grünbaum and

Haine [18,19,21] started a study of a discrete version of the original problem, by replacing L by a doubly infinite tridiagonal matrix. If one imposes special boundary conditions on the joint eigenfunctions, this problem contains the classical problem of classifying orthogonal polynomials which are eigenfunctions of a differential operator, and leads to extensions of the Askey–Wilson polynomials when B is a q-difference operator [4,10,20,22]. For a comprehensive review of the 'difference, differential (q-difference)' bispectral problem we refer the reader to the recent survey paper [23].

The 'q-difference, q-difference' version of the bispectral problem, that we study in this paper, can be looked up as a natural connection between the different bispectral situations. Indeed, any 'q-difference' operator $L(x, D_{q,x})$ can be considered as a difference operator if we pose $x = q^n$, with $n \in \mathbb{Z}$ and becomes a differential operator in the limit $q \to 1$ (if it exists). At present, it seems to offer the simplest instance among the various discrete versions of the bispectral problem, which can be solved for arbitrary order operators (at least in rank one). The q-deformed Grassmannian Gr_q^{ad}, that we construct, is still contained in the sub-Grassmannian Gr^{rat} which parametrizes the solutions of the KP hierarchy arising from rational algebraic curves. The intersection $Gr_q^{\mathrm{ad}} \cap Gr^{\mathrm{ad}}$ coincides with the sub-Grassmannian Gr_0 whose tau functions are polynomials in only finitely many time variables $t_1, t_2, \ldots$. As a consequence, the rational curves corresponding to planes $W \in Gr_q^{\mathrm{ad}} \backslash Gr_0$ must have *at least one node as a singular point*.

Using the correspondence between the q-tau functions and the planes in Sato Grassmannian, we construct a commutative algebra $\mathcal{A}_W^q$ of q-difference operators from any plane $W \in Gr$. For $W \in Gr_q^{\mathrm{ad}}$, the corresponding q-tau function $\tau_W^q(x, t)$ is a polynomial in x, which allows us to show in Section 3 the existence of a bispectral operator $B(z, D_{q,z})$ for any polynomial $\theta(x)$, such that $D_{q,x}\theta(x)$ is divisible by $\tau_W^q(xq)$. However, in contrast to the $q = 1$ case, for a generic plane $W \in Gr_q^{\mathrm{ad}}$, the tau function $\tau_W^q(x, t)$ is no longer polynomial in the time variables $t_1, t_2, \ldots$. In Section 5 we consider such a situation, which corresponds to a specific N-soliton solution. Formula (5.11) represents an extension of Shiota–Wilson formula for the rational KP solutions [40,42]. As an immediate consequence, we show that in this case the symmetry β in Gr^{ad} can be extended to Gr_q^{ad}. Moreover, as in the $q = 1$ case [42], β corresponds to a very simple involution at the level of Calogero–Moser pairs of matrices, see Theorem 5.3.

Finally, in Section 6, we examine the dynamics of the poles of the rational solutions (in x) to the q-KP hierarchy and show that the motion is governed by a hierarchy of Hamiltonian systems. The nth Hamiltonian, corresponding to the nth KP flow, is of the form

$$H_n = (-1)^n \frac{[n]_q}{n} \mathrm{Tr}(Y^n),$$

where Y is a deformation of the Calogero–Moser matrix, see Theorem 6.1. This result can be looked up as a q-analogue of the mysterious connection between the KP hierarchy and the Calogero–Moser hierarchy [3,32,40]. The derivation of the system (6.6) is obtained by a suitable adaptation of the approach of Shiota [40] for the classical case, within the context of the q-KP hierarchy. The main difficulty here, compared to the $q = 1$ case, comes from the non-triviality of the first q-KP flow. The key new ingredient is that $\partial/\partial t_1$ can be rewritten in a Lax form (see Lemma 6.2), which allows us to write the system in the Hamiltonian form above.

Some of the results in the present paper were announced in a brief note [26].

2. The q-KP hierarchy and algebras of q-difference operators

In this section we review briefly a q-analogue of the KP hierarchy, introduced in [24]. The exposition is based on a q-deformation of Sato theory [14,37] and follows closely [25]. For an alternative approach, using a correspondence with the Toda lattice hierarchy, we refer the reader to [1].

The q-derivative $D_{q,x} f$ of a function $f(x)$ is given by

$$(D_{q,x} f)(x) = \frac{f(xq) - f(x)}{x(q-1)}, \quad x \neq 0,$$

and $(D_{q,x} f)(0) = f'(0)$, by continuity, provided $f'(0)$ exists. We define $D^n_{q,x} \cdot f(x)$ for any $n \in \mathbb{Z}$, as the formal q-pseudo-difference operator

$$D^n_{q,x} \cdot f = \sum_{k=0}^{\infty} \binom{n}{k}_q (D^k_{q,x} f)(xq^{n-k}) D^{n-k}_{q,x},$$

where

$$\binom{n}{0}_q = 1, \qquad \binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Consider the formal q-pseudo-difference operator

$$L = D_{q,x} + a_0 + \sum_{i=1}^{\infty} a_i D^{-i}_{q,x}.$$

The q-deformed Kadomtsev–Petviashvili (in short q-KP) hierarchy is defined by the Lax equations

$$\frac{\partial L}{\partial t_j} = [(L^j)_+, L], \tag{2.1}$$

where $(L^j)_+$ denotes the positive part of the pseudo-difference operator L^j. One can define analogues of the wave function $\Psi^q(x, t_1, t_2, \ldots, z)$ and the tau function $\tau^q(x, t_1, t_2, \ldots)$, [2] which are connected by Sato formula

$$\begin{aligned}\Psi^q(x, t_1, t_2, \ldots, z) &= \frac{\tau^q(x, t_1 - 1/z, t_2 - 1/2z^2, \ldots)}{\tau^q(x, t_1, t_2, \ldots)} e_q^{xz} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \\ &= \psi^q(x, t_1, t_2, \ldots, z) e_q^{xz} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right),\end{aligned} \tag{2.2}$$

[2] It is a tau function in t in the sense of Sato theory for any x fixed.

where

$$e_q^x = \sum_{k=0}^{\infty} \frac{(1-q)^k}{(1-q)(1-q^2)\cdots(1-q^k)} x^k$$

denotes the q-exponential. The operator L is conjugated to $D_{q,x}$ by the wave operator $S =: \psi^q(x, t_1, t_2, \dots, D_{q,x}) :$ [3]

$$L = S D_{q,x} S^{-1}.$$

This formula allows us to express all functions $\{a_i(x,t)\}$ in terms of the tau function $\tau^q(x,t)$, where we have put $t = (t_1, t_2, \dots)$. In particular, we have

$$a_0(x,t) = \frac{\partial}{\partial t_1} \log \frac{\tau^q(xq,t)}{\tau^q(x,t)}. \tag{2.3}$$

The KP flows are represented on the wave function by the formulae

$$\frac{\partial \Psi^q}{\partial t_k} = (L^k)_+ \Psi^q \tag{2.4}$$

for $k = 1, 2, \dots$ and the operator L acts as a multiplication by z

$$L\Psi^q = z\Psi^q. \tag{2.5}$$

From (2.3) and (2.4) for $k = 1$ we get

$$\frac{\partial \Psi^q}{\partial t_1}(x,t,z) = \left(D_{q,x} + \frac{\partial}{\partial t_1} \log \frac{\tau^q(xq,t)}{\tau^q(x,t)} \right) \Psi^q(x,t,z). \tag{2.6}$$

The last equality, combined with the fact that $\tau^q(x,t)$ is a tau function in the sense of Kyoto school for any x fixed, characterizes completely the q-tau functions. The next theorem gives a simple explicit description of the q-tau functions in terms of the classical tau functions.

Theorem 2.1. *A formal power series $\tau^q(x,t) \in \mathbb{C}[[x, t_1, t_2, \dots]]$ is a tau function for the q-KP hierarchy if and only if, up to an unessential factor depending only on x, we have*

$$\tau^q(x,t) = \tilde{\tau}(t + [x]_q), \tag{2.7}$$

where $\tilde{\tau}(t) \in \mathbb{C}[[t_1, t_2, \dots]]$ is a tau function for the classical KP hierarchy, and

$$[x]_q = \left(x, \frac{(1-q)^2}{2(1-q^2)} x^2, \frac{(1-q)^3}{3(1-q^3)} x^3, \dots \right).$$

Proof. The 'if' part was proved in [1,25]. Below we prove the 'only if' part of the theorem. From (2.2) it is clear that if we multiply τ^q by a function which depends only on x, we get another tau function for the same solution. Thus, without any restriction, we may

[3] As usual : : means normal ordering, i.e. always pull the q-difference operator to the right.

suppose that $\tau^q(0,t)\not\equiv 0$. Plugging (2.2) in (2.6) and cancelling the exponential part we get

$$\left(z+\frac{1}{x(q-1)}\right)\left(\frac{\tau^q(xq,t-[z^{-1}])}{\tau^q(xq,t)}-\frac{\tau^q(x,t-[z^{-1}])}{\tau^q(x,t)}\right)$$
$$=\frac{\tau^q(x,t-[z^{-1}])}{\tau^q(x,t)}\left(\frac{\partial}{\partial t_1}\log\tau^q(x,t-[z^{-1}])-\frac{\partial}{\partial t_1}\log\tau^q(xq,t)\right), \tag{2.8}$$

where

$$[z]=[z]_0=\left(z,\frac{z^2}{2},\frac{z^3}{3},\dots\right).$$

If we put $z=-1/x(q-1)$ in the above identity we obtain

$$\frac{\partial}{\partial t_1}\log\tau^q(x,t-[x(1-q)])=\frac{\partial}{\partial t_1}\log\tau^q(xq,t),$$

and we can rewrite (2.8) as

$$\frac{\tau^q(xq,t-[z^{-1}])}{\tau^q(xq,t)}-\frac{\tau^q(x,t-[z^{-1}])}{\tau^q(x,t)}$$
$$=\left(z+\frac{1}{x(q-1)}\right)^{-1}\frac{\{\tau^q(x,t-[z^{-1}]),\tau^q(x,t-[x(1-q)])\}}{\tau^q(x,t)\tau^q(x,t-[x(1-q)])} \tag{2.9}$$

with

$$\{f,g\}:=\frac{\partial f}{\partial t_1}g-f\frac{\partial g}{\partial t_1}.$$

Since $\tau^q(x,t)$ is a classical tau function in $t_1,t_2,\dots$, it satisfies the differential Fay identity due to Adler and van Moerbeke [2]

$$\frac{\{\tau^q(x,t-[z^{-1}]),\tau^q(x,t-[y^{-1}])\}}{z-y}$$
$$=-\tau^q(x,t-[z^{-1}])\tau^q(x,t-[y^{-1}])+\tau^q(x,t)\tau^q(x,t-[z^{-1}]-[y^{-1}]). \tag{2.10}$$

For $y^{-1}=x(1-q)$ from (2.9) and (2.10) we get

$$\frac{\tau^q(xq,t-[z^{-1}])}{\tau^q(xq,t)}=\frac{\tau^q(x,t-[z^{-1}]-[x(1-q)])}{\tau^q(x,t-[x(1-q)])}. \tag{2.11}$$

Let us consider a new tau function $\tilde\tau(x,t):=\tau^q(x,t-[x]_q)$. Replacing t by $t-[xq]_q$ in (2.11) and using that $[xq]_q+[x(1-q)]=[x]_q$ we obtain

$$\frac{\tilde\tau(xq,t-[z^{-1}])}{\tilde\tau(xq,t)}=\frac{\tilde\tau(x,t-[z^{-1}])}{\tilde\tau(x,t)}.$$

The last equality simply means that the ratio $\tilde{\tau}(x, t-[z^{-1}])/\tilde{\tau}(x, t)$ does not depend on x, and so we have

$$\frac{\tilde{\tau}(x, t-[z^{-1}])}{\tilde{\tau}(0, t-[z^{-1}])} = \frac{\tilde{\tau}(x, t)}{\tilde{\tau}(0, t)}.$$

From this equation it follows that $\tilde{\tau}(x, t)/\tilde{\tau}(0, t) = f(x)$ does not depend on $t_1, t_2, \ldots$. Thus, we finally obtain

$$\tau^q(x, t) = f(x)\tilde{\tau}(0, t+[x]_q),$$

which finishes the proof of the theorem. □

Using this simple correspondence between the q-tau functions and the classical tau functions we can construct commutative algebras $\mathcal{A}_W^q$ of q-difference operators from any plane W from Sato Grassmannian. The q-wave function $\Psi_W^q(x, t, z) = \Psi_W(t+[x]_q, z)$ constructed in Theorem 2.1 can be characterized as the unique function $\Psi_W^q(x, t, z) \in W$ of the form

$$\Psi_W^q(x, t, z) = \left(1 + \sum_{i=1}^{\infty} \alpha_i(x, t) z^{-i}\right) e_q^{xz} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right).$$

Consider the algebra A_W of meromorphic functions $f(z)$ with poles only at $z = \infty$ that leave W invariant:

$$A_W = \{f(z) : f(z)W \subset W\}.$$

From the above characterization of the q-wave function $\Psi^q(x, t, z)$ and the definition of A_W, one can easily show that for any $f(z) \in A_W$, there exists a q-difference operator $L_f(x, t, D_{q,x})$ such that

$$L_f(x, t, D_{q,x})\Psi_W^q(x, t, z) = f(z)\Psi_W^q(x, t, z). \tag{2.12}$$

If L_W denotes the solution of the q-KP hierarchy, corresponding to the plane W, from (2.5) we can write the following 'explicit' formula for $L_f(x, t, D_{q,x})$:

$$L_f(x, t, D_{q,x}) = f(L_W). \tag{2.13}$$

Now, if we define

$$\mathcal{A}_W^q = \{L_f(x, 0, D_{q,x}) : f(z) \in A_W\},$$

we obtain a commutative algebra of q-difference operators isomorphic to A_W with common eigenfunction $\bar{\Psi}_W^q(x, z) = \Psi_W^q(x, 0, z)$. This algebra is non-trivial if W corresponds to an algebro-geometric solution of the KP hierarchy, see [31,34,38]. The spaces W arising from algebro-geometric data are precisely those such that A_W contains an element of any sufficiently large order. In the next section we shall use the above construction for the sub-Grassmannian Gr^{rat}, consisting of planes $W \in Gr$, corresponding to rational algebraic curves. [4] In this case $A_W \subset \mathbb{C}[z]$.

[4] Gr^{rat} is denoted by Gr_1 in [38].

3. The q-adelic Grassmannian Gr_q^{ad} and the bispectral problem

In this section we define Gr_q^{ad} and show the bispectrality of the corresponding algebras of q-difference operators. The proof is based on a q-version of the lemma due to Reach [36], which was used in [24] to prove the bispectral property of the q-deformed Schur polynomials. In the $q = 1$ case, this lemma was first explored by Zubelli [43], who showed the bispectral property of the classical Schur polynomials, and later by Liberati [33] who extended the construction to the adelic Grassmannian.

Inspired by Wilson [41], we consider the linear functionals (*q-conditions*) $e_q(m, \lambda)$ on $\mathbb{C}[z]$, defined by

$$\langle e_q(m, \lambda), g\rangle = (D^m_{q,z} g)(\lambda),$$

for $m \geq 0$ and $\lambda \in \mathbb{C}$. We denote by $\mathcal{C}^q_\lambda$ the infinite dimensional space over $\mathbb{C}$, generated by $e_q(m, \lambda)$ for $m \geq 0$, and by $\mathcal{C}^q$ the infinite dimensional space over $\mathbb{C}$, generated by all q-conditions. In contrast to the classical case, $e_q(m, \lambda)$ are no longer linearly independent. It is obvious from the definition that, for $\lambda \neq 0$,

$$\cdots \subset \mathcal{C}^q_{\lambda q^2} \subset \mathcal{C}^q_{\lambda q} \subset \mathcal{C}^q_\lambda \subset \mathcal{C}^q_{\lambda q^{-1}} \subset \mathcal{C}^q_{\lambda q^{-2}} \subset \cdots .$$

A functional c is called a *one point q-condition* if it is a finite linear combination of q-conditions supported at single point λ, i.e. $c \in \mathcal{C}^q_\lambda$. For each finite dimensional subspace $C \subset \mathcal{C}^q$, we set

$$V_C = \{g(z) \in \mathbb{C}[z] : \langle c, g\rangle = 0 \quad \text{for} \quad c \in C\}.$$

Now we are ready to give the definition of the q-deformed adelic Grassmannian Gr_q^{ad}.

Definition 3.1. A plane $W \in Gr$ belongs to Gr_q^{ad} if W has the form $W = r^{-1}(z)V_C$, for some finite dimensional subspace $C \subset \mathcal{C}^q$, which possesses a basis of one-point q-conditions, and $r(z)$ is the unique polynomial in z of degree $\deg r(z) = \dim C$, such that

$$\lim_{x\to\infty} \psi^q_W|_{t=0} = 1.$$

Remark 3.2. *From the definition it follows directly that Gr_q^{ad} is contained in the Grassmannian Gr^{rat}, which corresponds to the algebro-geometric solutions of the KP hierarchy, arising from rational algebraic curves, see* [41]. *In particular, for any $W \in Gr_q^{\mathrm{ad}}$,* Spec A_W *is a rational curve. The intersection $Gr^{\mathrm{ad}} \cap Gr_q^{\mathrm{ad}}$ is the sub-Grassmannian Gr_0, corresponding to planes $W \in Gr$, with a tau function $\tau_W(t)$ polynomial in a finite number of the time variables $t_1, t_2, \ldots$.*

Remark 3.3. *The group Γ_- of rational functions $\gamma(z)$ with $\gamma(\infty) = 1$ acts*[5] *on Gr^{rat} by scalar multiplication and the q-wave function of $\gamma(z)W$ is just $\gamma(z)\Psi^q_W$. Thus, the algebra*

[5] These transformations are sometimes referred to as *gauge transformations*.

$\mathcal{A}_W^q$ constructed from W depends only on the Γ_- -orbit in Gr^{rat}, which gives us some freedom in choosing $r(z)$. The special choice of $r(z)$ above is made to fix the plane in each orbit of Γ_-, whose tau function is, up to an unessential factor, a polynomial in x with constant leading coefficient (*i.e. it can be taken to be a monic polynomial*). *This normalization is used for the extension of the bispectral involution in Section* 5. *The explicit formula for $r(z)$ will be computed later* (*see* (3.7)).

Let us fix a plane $W = r^{-1}(z)V_C \in Gr_q^{\mathrm{ad}}$ with $C = \{c_1, c_2, \ldots, c_N\}$ as in the definition. Since $\{c_i\}$ are one point q-conditions we can write

$$c_k = \sum_{i=1}^{s_k} \gamma_{ki}\, e_q(i, \lambda_k),$$

where s_k is the order [6] of the condition c_k. From the characterization of the q-wave function in the previous section and the definition of W, one obtains the following explicit formula for $\Psi_W^q(x, t, z)$:

$$\Psi_W^q(x, t, z) = \frac{1}{r(z)} \frac{\mathrm{Wr}_q(f_1, f_2, \ldots, f_N, e_q^{xz})}{\mathrm{Wr}_q(f_1, f_2, \ldots, f_N)} \exp\left(\sum_{k=1}^{\infty} t_k z^k\right), \tag{3.1}$$

where $f_k(x,t) = \left\langle c_k, e_q^{xz} \exp\left(\sum t_i z^i\right)\right\rangle$, and $\mathrm{Wr}_q(f_1, \ldots, f_N)$ denotes the q-Wronskian determinant $\det(D_{q,x}^{i-1} f_j)$. From the defining relation of $\{f_k\}$, it is not difficult to check that they satisfy

$$f_k(x, t - [z^{-1}]) = f_k(x,t) - \frac{1}{z} D_{q,x} f_k(x,t). \tag{3.2}$$

Using (3.1) and (3.2) and the elementary properties of determinants we can rewrite $\Psi_W^q(x, t, z)$ in the form

$$\Psi_W^q(x, t, z) = \frac{z^N}{r(z)} \frac{\mathrm{Wr}_q(f_1(x, t-[z^{-1}]), \ldots, f_N(x, t-[z^{-1}]))}{\mathrm{Wr}_q(f_1(x,t), \ldots, f_N(x,t))} e_q^{xz} \exp\left(\sum_{k=1}^{\infty} t_k z^k\right). \tag{3.3}$$

From the last equality, it follows that $\tau_W^q(x, t)$ is a polynomial in x given by

$$\tau_W^q(x,t) = \mathrm{Wr}_q(f_1, f_2, \ldots, f_N) \left(e_q^{\lambda_1 x} \cdots e_q^{\lambda_N x}\right)^{-1} \exp\left(\sum_{i=1}^{\infty} \beta_i t_i\right), \tag{3.4}$$

where $\{\beta_i\}$ are constants determined by the equality

$$\frac{r(z)}{z^N} = \exp\left(\sum_{i=1}^{\infty} \frac{\beta_i}{iz^i}\right).$$

[6] One should be careful here because the same condition can be written as a condition at the point $\lambda_k q^{-m}$ and then the order will be $s_k + m$.

Substituting $t_1 = t_2 = \cdots = 0$ in (3.1) and (3.4) we obtain

$$\bar{\tau}^q_W(x) := \tau^q_W(x,0) = \mathrm{Wr}_q(p_1(x)e_q^{\lambda_1 x}, \dots, p_N(x)e_q^{\lambda_N x})(e_q^{\lambda_1 x}\cdots e_q^{\lambda_N x})^{-1}, \tag{3.5}$$

$$\bar{\Psi}^q_W(x,z) = \Psi^q_W(x,0,z) = \frac{\mathrm{Wr}_q(p_1(x)e_q^{\lambda_1 x}, \dots, p_N(x)e_q^{\lambda_N x}, e_q^{xz})}{r(z)\tau^q_W(x,0)e_q^{\lambda_1 x}\cdots e_q^{\lambda_N x}}, \tag{3.6}$$

where $p_k(x) = f_k(x,0)(e_q^{\lambda_k x})^{-1} = \sum_{i=1}^{s_k}\gamma_{ki}x^i$ for $k = 1, 2, \dots, N$ and $\bar{\tau}^q_W(x)$ are polynomials in x.

To write an explicit formula for $r(z)$ we shall suppose that $\lambda_i q^{s_i} \neq \lambda_j q^{s_j}$ for $i \neq j$. This inequality can always be achieved by picking an appropriate basis of C. Indeed, $\lambda_i q^{s_i} = \lambda_j q^{s_j}$ means that c_i and c_j can be looked up as conditions of the same order supported at the point $\lambda_i q^{-s}$ for some s big enough; taking appropriate linear combinations we may assume that this never happens. Now, in the limit $x \to \infty$, from (3.5) and (3.6) one can deduce that

$$r(z) = \prod_{k=1}^{N}(z - \lambda_k q^{s_k}). \tag{3.7}$$

Let us now formulate a q-analogue of the lemma due to Reach [36].

Lemma 3.4. *Let $g_0, g_1, \dots, g_{N+1}$ be functions of x. Define*

$$G(x) = \sum_{k=1}^{N+1}(-1)^{N+1+k}g_k(x)\int g_0(x)\mathrm{Wr}_q(g_1, \dots, \hat{g}_k, \dots, g_{N+1})\,\mathrm{d}_q x. \tag{3.8}$$

Then

$$\mathrm{Wr}_q(g_1, g_2, \dots, g_N, G) = \theta(x)\,\mathrm{Wr}_q(g_1, g_2, \dots, g_{N+1}) \tag{3.9}$$

with

$$\theta(x) = \left.\left(\int g_0(x)\,\mathrm{Wr}_q(g_1, g_2, \dots, g_N)\,\mathrm{d}_q x\right)\right|_{xq}, \tag{3.10}$$

where $\int \mathrm{d}_q x$ denotes the standard q-integral.

The proof of this simple but important lemma can be found in [24]. We can now state the main result of this section.

Theorem 3.5. *For each plane $W \in Gr_q^{\mathrm{ad}}$ the commutative algebra of q-difference operators $\mathcal{A}^q_W$ is bispectral. Precisely, the function $\bar{\Psi}^q_W(x,z)$ satisfies*

$$L_f(x, D_{q,x})\bar{\Psi}^q_W(x,z) = f(z)\bar{\Psi}^q_W(x,z) \tag{3.11}$$

for $f(z) \in A_W$, and if $\theta(x)$ is a polynomial in x such that $D_{q,x}\theta(x)$ is divisible by $\bar{\tau}^q_W(xq)$, there exists a q-difference operator in z, $B_\theta(z, D_{q,z})$ independent of x such that

$$B_\theta(z, D_{q,z})\bar{\Psi}^q_W(x,z) = \theta(x)\bar{\Psi}^q_W(x,z). \tag{3.12}$$

Proof. By q-integration by parts, for any polynomial $h(x)$, we have

$$\int h(x)\, e_q^{xz}(e_q^{\lambda x})^{-1}\,\mathrm{d}_q x = -\sum_{k=0}^{\infty}\frac{(\lambda(q-1)x+q)(\lambda(q-1)x+q^2)\cdots(\lambda(q-1)x+q^{k+1})}{q^{k(k+1)/2}(\lambda-qz)(\lambda-q^2z)\cdots(\lambda-q^{k+1}z)} \times (D_{q,x}^k h)\left(\frac{x}{q^{k+1}}\right) e_q^{xz}(e_q^{\lambda x})^{-1}. \tag{3.13}$$

Now we apply Lemma 3.4 with $g_0(x) = p(x)\prod_{i=1}^{N}(e_q^{\lambda_i x})^{-1}$, where $p(x)$ is a polynomial in x, $g_i(x) = p_i(x)e_q^{\lambda_i x}$ for $i = 1, 2, \dots, N$, and $g_{N+1} = e_q^{xz}/r(z)$. Using (3.13) we see that G can be written as

$$G = P(x,z)e_q^{xz},$$

where $P(x,z)$ is a polynomial in x with rational in z coefficients. Thus, replacing $x\, e_q^{xz}$ by $D_{q,z}\, e_q^{xz}$ we get

$$G =: P(D_{q,z}, z) : e_q^{xz} = B(z, D_{q,z})\frac{e_q^{xz}}{r(z)}. \tag{3.14}$$

Putting (3.14) into (3.9) and using (3.5) and (3.6), we obtain

$$B(z, D_{q,z})\bar{\Psi}_W^q(x,z) = \theta(x)\bar{\Psi}_W^q(x,z)$$

with

$$\theta(x) = \left(\int p(x)\bar{\tau}_W^q(x)\,\mathrm{d}_q x\right)\Big|_{xq},$$

from which it follows that $\theta(x)$ can be any polynomial in x such that $D_{q,x}\theta(x)$ is divisible by $\bar{\tau}_W^q(xq)$. □

We shall illustrate all steps of the above construction in the next section.

4. Some elementary examples

In this section we present a few simple examples of bispectral algebras of q-difference operators.

Example 4.1. As a first example let us take W to be the space of $Gr_0 = Gr^{\mathrm{ad}} \cap Gr_q^{\mathrm{ad}}$ determined by the single condition $\langle c, g\rangle = g''(0) - \alpha g'(0)$, where α is some parameter. This corresponds to a situation slightly more complicated from the one considered in [24] with a tau function which is a finite linear combination of Schur polynomials. The curve Spec A_W has just one cusp at the origin. In fact we have $A_W = \mathbb{C}[z^3, z^4, z^5]$, so the singularity at zero is not planar. Since $z^3 \in A_W$, the q-pseudo-difference operator L_W solves the third

Gelfand–Dickey (or the third KdV) hierarchy, i.e. $\tilde{L} = L_W^3$ is a q-difference operator. From (3.4) the q-tau function for the corresponding plane

$$W = \frac{1}{z}\{g(z) \in \mathbb{C}[z] : g''(0) - \alpha g'(0) = 0\}$$

is

$$\tau_W^q(x,t) = \frac{2}{1+q}x^2 + (2t_1 - \alpha)x + t_1^2 - \alpha t_1 + 2t_2.$$

Hence

$$\bar{\tau}_W^q(x) = \frac{2}{1+q}x^2 - \alpha x.$$

The wave function, computed at $t_1 = t_2 = \cdots = 0$ is given by formula (3.6)

$$\bar{\Psi}_W^q(x,z) = \left(1 + \frac{(q+1)(\alpha - 2x)}{z(2x^2 - \alpha(q+1)x)}\right) e_q^{xz}.$$

If we take $f(z) = z^3 \in A_W$ one computes that

$$\begin{aligned}\tilde{L}=L_f = D_{q,x}^3 &+ \frac{(1-q^3)(1+q)(4q^3x^2 - 2\alpha(q^3+1)x + \alpha^2(q+1))}{q^3x(2x-\alpha(q+1))(2q^3x - \alpha(q+1))}D_{q,x}^2 \\ &- \frac{(q+1)(q^2+q+1)}{q^4x^2} \\ &\times \frac{(8q^6x^3 - 4\alpha q^3(q^2+1)(q+1)x^2 + 2\alpha^2(2q^3+1)(q+1)x - \alpha^3(q+1)^2)}{(2x-\alpha(q+1))(2q^2x-\alpha(q+1))(2q^3x-\alpha(q+1))}D_{q,x} \\ &+ \frac{\alpha^2(q+1)^3(q^2+q+1)(2(q^2-q+1)x - \alpha)}{q^4x^3(2x-\alpha(q+1))(2q^2x-\alpha(q+1))(2q^3x-\alpha(q+1))}.\end{aligned}$$

We choose

$$\theta(x) = x^3 - \frac{\alpha}{2q}(q^2+q+1)x^2$$

such that

$$D_{q,x}\theta(x) = \frac{(q^2+q+1)(1+q)}{2q^2}\bar{\tau}_W^q(xq).$$

The bispectral operator $B_\theta(z, D_{q,z})$ is given by the formula

$$\begin{aligned}B_\theta(z, D_{q,z}) = D_{q,z}^3 &- \frac{(q^2+q+1)(\alpha q^2 z + 2q^2 - 2)}{2q^3 z}D_{q,z}^2 \\ &+ \frac{(q+1)(q^2+q+1)(\alpha(q-1)z - 2)}{2q^3z^2}D_{q,z} \\ &+ \frac{\alpha(q+1)(q^2+q+1)}{2q^3z^2}.\end{aligned}$$

The next example deals with the simplest possible plane W belonging to Gr_q^{ad} but not to Gr^{ad}.

Example 4.2. Let $C = \mathbb{C}\, e_q(1,1)$ be the space generated by the single condition $e_q(1,1)$ at the point $z = 1$. For $q \neq 1$ this is the simplest example of soliton solution of the KP hierarchy, consisting of a solitary wave. We have that $r(z) = z - q$ and the q-tau function for the corresponding plane $W = r^{-1}V_C$ is given by formula (3.4)

$$\tau_W^q(x,t) = x + \frac{\exp\left(\sum_{i=1}^{\infty} t_i q^i\right) - \exp\left(\sum_{i=1}^{\infty} t_i\right)}{q-1}.$$

Thus

$$\bar{\tau}_W^q(x) = x.$$

The wave function, computed at $t_1 = t_2 = t_3 = \cdots = 0$, is

$$\bar{\Psi}_W^q(x,z) = \left(1 - \frac{1}{x(z-q)}\right) e_q^{xz}.$$

The algebra A_W is generated by

$$f(z) = z^2 - (q+1)z + q$$

and

$$h(z) = z^3 - \tfrac{3}{2}(q+1)z^2 + \tfrac{1}{2}(q^2+4q+1)z - \tfrac{1}{2}(q^2+q),$$

where

$$L_f = D_{q,x}^2 - \frac{(q+1)(q^2x+q-1)}{q^2x} D_{q,x} + \frac{(q^3x^2-q-1)}{q^2x^2}.$$

If we choose for example $\theta = x^2$, the bispectral operator $B_\theta(z, D_{q,z})$ becomes

$$B_\theta(z, D_{q,z}) = D_{q,z}^2 + \frac{(1-q^2)z}{q(z-q)(zq-1)} D_{q,z} - \frac{q+1}{q(z-q)(zq-1)}.$$

Let us take $\xi = f(z)$ and $\eta = h(z)$ as generators of the coordinate ring. The corresponding curve is

$$\eta^2 = \xi^3 + \left(\frac{q-1}{2}\right)^2 \xi^2.$$

The sole singularity is a double point at the origin which becomes a cusp in the limit $q \to 1$, in agreement with Wilson's result.

Remark 4.3. *Using the above examples one can easily understand the picture in general. For a plane from the 'non-deformed' part* $Gr_0 = Gr^{\mathrm{ad}} \cap Gr_q^{\mathrm{ad}}$, *the tau function is a polynomial in finitely many time variables and we have a rational solution of the q-KP hierarchy* (*in all variables*). *In terms of Darboux transformation* (*see* [5,6,33]) *this corresponds to* *bispectral operators obtained by factorizing some powers of $D_{q,x}$. In the case of Example* 4.1, *the operator $\tilde{L} = L_f$ is a Darboux transformation from the operator $f(D_{q,x}) = D_{q,x}^3$.*

When we have conditions supported 'outside 0', we get soliton-like solutions of the q-KP hierarchy, which are rational in x, and which are rational in all time-variables in the limit $q \to 1$. This corresponds to some deformation of the constant coefficient operator from which we do the Darboux transformation. In Example 4.2 *the operator L_f is a Darboux transformation from the second order q-difference operator*

$$f(D_{q,x}) = (D_{q,x} - 1)(D_{q,x} - q).$$

Note also that the operator $(D_{q,x} - 1)^2$ cannot be 'rationally' factorized in a different way. In general, one would easily show that the bispectral operators parametrized by Gr_q^{ad} can be described as Darboux transformations from constant coefficient q-difference operators of the form

$$L = \prod_{i=1}^{N}\prod_{j=1}^{k_i}(D_{q,x} - \lambda_i q^{j-1}).$$

In the next section, we consider the 'generic' case of planes generated by first order conditions at different points and show that in this case the bispectral property is related to a symmetry in Gr_q^{ad}. In the case $q = 1$, this is the clue to the connection with the Calogero–Moser system (see [27,42]).

5. q-Calogero–Moser matrix and the bispectral problem

Let us take first order conditions $\{c_1, \dots, c_N\}$, supported at different points $\{\lambda_1, \dots, \lambda_N\}$, which satisfy also[7] $\lambda_i \neq q\lambda_j$ for $i \neq j$, i.e. $c_i = e_q(1, \lambda_i) + \alpha_i e_q(0, \lambda_i)$ and consider the plane

$$W = \prod_{j=1}^{N}(z - q\lambda_j)^{-1} V_C. \tag{5.1}$$

Let us denote by Λ and α the diagonal matrices $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ and $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_N)$, and by $\mathrm{Van}(\lambda) = (\lambda_i^{j-1})_{1\le i,j\le N}$ — the Vandermonde matrix. Now, using (3.4) one can write the tau function in the form

$$\tau_W^q(x,t) = \det\Bigg[x\,\mathrm{Van}(q\lambda) + V + \mathrm{Van}(\lambda)\left(\alpha \exp\left(\sum t_i(1-q^i)\Lambda^i\right) + \frac{\exp\left(\sum t_i(1-q^i)\Lambda^i\right) - E}{1-q}\Lambda^{-1}\right)\Bigg], \tag{5.2}$$

---

[7] In the rest of the paper we shall briefly call $\{\lambda_i\}$ 'q-different' if they are different and $\lambda_i \neq q\lambda_j$ for $i \neq j$.

where $\mathrm{Van}(q\lambda) = \mathrm{Van}(q\lambda_1, q\lambda_2, \dots, q\lambda_N), E = E_N$ is the identity $N \times N$ matrix and

$$V = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ [2]_q\,\lambda_1 & [2]_q\,\lambda_2 & \dots & [2]_q\,\lambda_N \\ \vdots & \vdots & & \vdots \\ [N-1]_q\,\lambda_1^{N-2} & [N-1]_q\,\lambda_2^{N-2} & \dots & [N-1]_q\,\lambda_N^{N-2} \end{pmatrix}$$

with

$$[n]_q = \frac{1-q^n}{1-q}.$$

We shall also need two diagonal matrices $A(\lambda) = \mathrm{diag}(A_1, A_2, \dots, A_N)$ and $A'(\lambda) = \mathrm{diag}(A'_1, A'_2, \dots, A'_N)$ defined by

$$A_i = \prod_{j\neq i} \frac{q\lambda_i - \lambda_j}{\lambda_i - \lambda_j}, \qquad A'_i = \prod_{j\neq i} (q\lambda_i - \lambda_j).$$

With these notations one can check that V can be written as

$$V = -\mathrm{Van}(\lambda) A'^{-1} \tilde{\Lambda} A', \tag{5.3}$$

where $\tilde{\Lambda}$ is a matrix of Calogero–Moser type

$$\tilde{\Lambda}_{ij} = \frac{A_i(\lambda)}{\lambda_i - q\lambda_j} \quad \text{for } i \neq j, \tag{5.4}$$

$$\tilde{\Lambda}_{ii} = \frac{1 - A_i(\lambda)}{\lambda_i(q-1)}. \tag{5.5}$$

From (5.3) it easily follows that

$$\mathrm{Van}(q\lambda) = \mathrm{Van}(\lambda) A'^{-1} (E + (1-q)\tilde{\Lambda}\Lambda) A'. \tag{5.6}$$

Now, using (5.2),(5.3) and (5.6) and the fact that A', Λ and α are diagonal matrices (and so, in particular, they commute) we get the following formula for τ_W^q:

$$\tau_W^q = \det(\mathrm{Van}(\lambda)) \det\left(x(E + (1-q)\tilde{\Lambda}\Lambda) - \tilde{\Lambda} + \alpha \exp\left(\sum t_i (1-q^i)\Lambda^i\right) \right. \\ \left. + \frac{\exp\left(\sum t_i(1-q^i)\Lambda^i\right) - E}{1-q} \Lambda^{-1} \right). \tag{5.7}$$

From (5.6) it follows that

$$\det(E + (1-q)\tilde{\Lambda}\Lambda) = q^{N(N-1)/2} \neq 0,$$

thus we can define

$$X_t = (E + (1-q)\tilde{\Lambda}\Lambda)^{-1}\left(\tilde{\Lambda} - \alpha \exp\left(\sum t_i(1-q^i)\Lambda^i\right)\right.$$
$$\left. + \frac{E - \exp\left(\sum t_i(1-q^i)\Lambda^i\right)}{1-q}\Lambda^{-1}\right). \quad (5.8)$$

In particular, for $t = 0$, we have

$$X_0 = (E + (1-q)\tilde{\Lambda}\Lambda)^{-1}(\tilde{\Lambda} - \alpha). \quad (5.9)$$

X_t and X_0 are connected by

$$X_t = X_0 \exp\left(\sum t_i(1-q^i)\Lambda^i\right) + \frac{E - \exp\left(\sum t_i(1-q^i)\Lambda^i\right)}{1-q}\Lambda^{-1}. \quad (5.10)$$

From the last equality we get

$$X_{t-[z^{-1}]} = (X_t(zE - \Lambda) + E)(zE - q\Lambda)^{-1},$$

and thus we can finally write explicit formulae (cf. [40,42]) for the tau function and the wave function

$$\tau_W^q(x,t) = \det(\mathrm{Van}(q\lambda))\det\left(xE - X_0 \exp\left(\sum t_i(1-q^i)\Lambda^i\right)\right.$$
$$\left. + \frac{\exp\left(\sum t_i(1-q^i)\Lambda^i\right) - E}{1-q}\Lambda^{-1}\right), \quad (5.11)$$

$$\Psi_W^q(x,t,z) = \frac{\det(xzE - xq\Lambda - zX_t + X_t\Lambda - E)}{\det(xE - X_t)\det(zE - q\Lambda)} \mathrm{e}_q^{xz} \exp\left(\sum_{k=1}^{\infty} t_k z^k\right). \quad (5.12)$$

One can check that the matrices Λ, $\tilde{\Lambda}$ and X_t satisfy the following relations:

$$[\Lambda, \tilde{\Lambda}]_q + E = AT, \quad [X_t, \Lambda]_q - E = ([X_0, \Lambda] - E)\exp\left(\sum t_i(1-q^i)\Lambda^i\right),$$
$$[X_0, \Lambda]_q - E = -(E + (1-q)\tilde{\Lambda}\Lambda)^{-1}AT(E + (q-1)\Lambda X_0),$$

where $[P, Q]_q = PQ - qQP$ denotes the q-commutator and

$$T = T_N = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}.$$

In particular, we have

$$\mathrm{rank}([X_t, \Lambda]_q - E) = 1.$$

Our next goal will be to extend the symmetry in the adelic Grassmannian to the $q \neq 1$ case for a generic plane $W \in Gr_q^{\mathrm{ad}}$ of the form (5.1). Before that, we shall formulate a technical lemma.

Lemma 5.1. *In the notations above, the identity*

$$(E+(1-q)\tilde{\Lambda}\Lambda)AT = q^{N-1}AT \tag{5.13}$$

holds.

Proof. Equality (5.13) is equivalent to

$$\sum_{k=1}^{N}\frac{\lambda_k}{\lambda_s - q\lambda_k}\prod_{i\neq k}\frac{q\lambda_k-\lambda_i}{\lambda_k-\lambda_i} = \frac{q^{N-1}}{1-q}. \tag{5.14}$$

The left-hand side of (5.14) can be rewritten as

$$\frac{1}{\det(\mathrm{Van}(\lambda))}\left[\sum_{k=1}^{N}\frac{\lambda_k}{\lambda_s-q\lambda_k}\det(\mathrm{Van}(\lambda_1,\lambda_2,\dots,q\lambda_k,\dots,\lambda_N))\right] = \frac{F(\lambda)}{\det(\mathrm{Van}(\lambda))}.$$

$F(\lambda)$ is a polynomial in $\{\lambda_i\}$ which is zero for $\lambda_i=\lambda_j$, hence $\det(\mathrm{Van}(\lambda))/F(\lambda)$. But since $\det(\mathrm{Van}(\lambda))$ and $F(\lambda)$ have the same degree, it follows that the left-hand side of (5.14) is a constant, which depends only on q and N. Taking $\lambda_s\to 0$, (5.14) reduces to

$$\sum_{k\neq s}\prod_{i\neq k,s}\frac{q\lambda_k-\lambda_i}{\lambda_k-\lambda_i} = [N-1]_q.$$

Remembering that the left-hand side does not depend on $\{\lambda_i\}$, one can easily prove the last equality by induction. □

Now, we are ready to characterize the planes of the form (5.1) by the next proposition.

Proposition 5.2. *Let X and Y be two $n\times n$ matrices, such that the eigenvalues of Y are q-distinct and*

$$\mathrm{rank}([X,Y]_q+E_n)=1.$$

Then, there exists $N\le n$ and a matrix $U\in GL(n,\mathbb{C})$ such that

$$Y=-U\,\mathrm{diag}(\underbrace{\lambda_1,\dots,\lambda_N}_{\Lambda},\underbrace{\lambda_{N+1},\dots,\lambda_n}_{\Lambda'})U^{-1} = -U\begin{pmatrix}\Lambda & 0\\ 0 & \Lambda'\end{pmatrix}U^{-1},$$

and X can be written in the block form

$$X=U\begin{pmatrix}X_0 & *\\ 0 & \Lambda''\end{pmatrix}U^{-1},$$

with $X_0=(E_N+(1-q)\tilde{\Lambda}\Lambda)^{-1}(\tilde{\Lambda}-\alpha)$, the $N\times N$ matrix given by (5.9) *for some diagonal matrix α, and $\Lambda''=((1-q)\Lambda')^{-1}$. Defining a plane $W=W(X,Y)\in Gr_q^{\mathrm{ad}}$ by* (5.1), *its wave function at $t=0$ is given by*

$$\bar{\Psi}_W^q(x,z)=\bar{\Psi}^q(x,z,X,Y)=\frac{\det(xzE_n+xqY-zX-XY-E_n)}{\det(xE_n-X)\det(zE_n+qY)}e_q^{xz}.$$

Proof. We can diagonalize the matrix Y by a matrix U_1 and write

$$[U_1^{-1}XU_1, \mathrm{diag}(\lambda_1, \ldots, \lambda_n)]_q - E_n = S'T_nS'',$$

where $Y = -U_1 \mathrm{diag}(\lambda_1, \ldots, \lambda_n)U_1^{-1}$, and S' and S'' are diagonal matrices (we still have the freedom to conjugate by diagonal matrices). Suppose that $S'_i \neq 0$ for $i = 1, 2, \ldots, N$ and $S'_i = 0$ for $i = N + 1, \ldots, n$. Then X is of the form

$$X = U_1 \begin{pmatrix} X' & * \\ 0 & \Lambda'' \end{pmatrix} U_1^{-1},$$

and conjugating by diagonal matrices we can make S'_i (for $i = 1, \ldots, N$) as arbitrary non-zero numbers. Let us fix $S'_i = -A_i(\lambda)$. Thus we get

$$[X', \Lambda]_q - E_N = -AT_N S \tag{5.15}$$

for some $N \times N$ matrix S. If we put

$$\alpha = \tilde{\Lambda} - (E_N + (1 - q)\tilde{\Lambda}\Lambda)X',$$

and multiply (5.15) to the left by $(E_N + (1 - q)\tilde{\Lambda}\Lambda)$, using Lemma 5.1, we obtain

$$[\Lambda, \alpha] = AT_N(E_N + (q - 1)\Lambda X' - q^{N-1}S).$$

From the last equality it follows that α is a diagonal matrix. Since

$$\bar{\Psi}^q(x, z, X, Y) = \bar{\Psi}^q(x, z, X', -\Lambda),$$

the rest of the argument is clear from (5.12). □

As an immediate corollary of Proposition 5.2, we can state the main result of this section.

Theorem 5.3. *Let X and Y be $n \times n$ matrices which have q-different eigenvalues and satisfy*

$$\mathrm{rank}([X, Y]_q + E_n) = 1.$$

Let $W = W(X, Y)$ and $W' = W'(-qY^t, -q^{-1}X^t)$ denote the planes constructed in Proposition 5.2. *Then we have*

$$\bar{\Psi}^q_W(x, z) = \bar{\Psi}^q_{W'}(z, x),$$

i.e., on pair of matrices, the bispectral involution corresponds to the map

$$\beta : (X, Y) \to (-qY^t, -q^{-1}X^t).$$

Remark 5.4. *Following Wilson* [42], *let us denote by V_n the complex vector space of pairs (X, Y), where X and Y are $n \times n$ matrices, and by $\tilde{C}^q_n$ the sub-variety of V_n, consisting of all (X, Y) satisfying the equation*

$$\mathrm{rank}([X, Y]_q + E_n) = 1. \tag{5.16}$$

The group $GL(n,\mathbb{C})$ acts on V_n by simultaneous conjugation of X and Y. Clearly this action preserves (5.16). *Let C_n^q stand for the quotient space $\tilde{C}_n^q/GL(n,\mathbb{C})$. Formula* (5.10) *suggests to introduce q-analogues of the Calogero–Moser flows on C_n^q, induced by the $GL(n,\mathbb{C})$ invariant flows on $\tilde{C}_n^q$*

$$(X,Y)\to\left(X\exp\left(\sum t_i(1-q^i)(-Y)^i\right)+\frac{E-\exp\left(\sum t_i(1-q^i)(-Y)^i\right)}{1-q}(-Y)^{-1},Y\right). \tag{5.17}$$

One can check that the above formula defines properly commutative flows on V_n, which preserve the condition (5.16). *However, in the case $q\neq 1$ these flows are not Hamiltonian* (*in the standard coordinates*), *and thus the reduction procedure of* [29] *cannot be easily applied. In the next section, we shall write the corresponding dynamical system on the reduced phase space in a Hamiltonian form, using the approach of Shiota* [40].

6. Rational solutions to q-KP hierarchy and the corresponding deformation of Calogero–Moser hierarchy

In [3] Airault et al. discovered an amazing relation between equations of KdV type and the Calogero–Moser system. Namely, they showed that the poles of a rational solution to the KdV or Boussinesq equation that vanishes at infinity is described by the Calogero–Moser system with inverse square potential, with some constraint on the configuration of poles. Krichever [32] observed that the poles of the rational solutions of the KP equation that vanish at $x=\infty$, move according to the Calogero–Moser system with no constraint and wrote down explicit formulae for these solutions. Finally, Shiota [40] extended this phenomenon to the whole KP hierarchy. The aim of the present section is to find the system of equations for the poles of the rational solutions to the q-KP hierarchy. Since the poles come from zeros of the tau function, let

$$\tau^q(x,t)=(x-x_1(t))(x-x_2(t))\cdots(x-x_N(t)) \tag{6.1}$$

be a q-tau function of the KP hierarchy, which is a polynomial in x. We may assume that $\partial x_j/\partial t_1\neq 0$ for $j=1,2,\dots,N$. Indeed, if $\partial x_j/\partial t_1=0$ for some j, then it is not difficult to see that $\partial x_j/\partial t_n=0$ for any n (cf. (6.9),(6.10) and (6.12) below), which means that $(x-x_j)$ is just an unessential factor in the tau function. We shall suppose also that $x_i\neq q^k x_j$ for $i\neq j$ and $k=0,1$. This is a natural restriction since in the limit $q\to 1$ it reduces to $x_i\neq x_j$ for $i\neq j$, which can always be achieved by picking an appropriate neighbourhood of the $\{t_i\}$'s, see [39,40]. Let us denote by $A_i=A_i(x_1,x_2,\dots,x_N)$ the expression from the previous section

$$A_i=\prod_{j\neq i}\frac{qx_i-x_j}{x_i-x_j} \tag{6.2}$$

and introduce $y_1, y_2, \ldots, y_N$ by

$$-x_i' = A_i\, \mathrm{e}^{(1-q)x_i y_i}, \tag{6.3}$$

where for simplicity we have posed $' = \partial/\partial t_1$. We define the q-deformed Calogero–Moser matrix Y by

$$Y_{ij} = -\frac{x_i'}{x_i - qx_j} = \frac{A_i\, \mathrm{e}^{(1-q)x_i y_i}}{x_i - qx_j}, \quad \text{for } i \neq j, \tag{6.4}$$

$$Y_{ii} = \frac{1 + x_i'}{x_i(q-1)} = \frac{1 - A_i\, \mathrm{e}^{(1-q)x_i y_i}}{x_i(q-1)}. \tag{6.5}$$

With these notations we can state the main result of this section.

Theorem 6.1. *Let $\tau^q(x,t) = \prod_{i=1}^{N}(x - x_i(t))$ be a tau function of the q-KP hierarchy, which is a monic polynomial in x. Then the motion of the zeros of τ^q is governed by a hierarchy of Hamiltonian systems, which is a q-deformation of the Calogero–Moser hierarchy. Precisely, if we define*

$$H_n = (-1)^n \frac{[n]_q}{n} \operatorname{tr}(Y^n),$$

we have

$$\frac{\partial}{\partial t_n}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \partial H_n/\partial y_i \\ -\partial H_n/\partial x_i \end{pmatrix}, \qquad n = 1, 2, \ldots \tag{6.6}$$

Proof. Consider the wave function

$$\Psi^q(x,t,z) = \left(\sum_{k=0}^{\infty} \psi_k z^{-k}\right) e_q^{xz} \exp\left(\sum_{k=1}^{\infty} t_k z^k\right), \tag{6.7}$$

where $\psi_0 = 1$ and ψ_k, $k > 0$, is given by

$$\psi_k = \frac{p_k(-\tilde{\partial})\tau^q}{\tau^q}. \tag{6.8}$$

For our special choice (6.1) of the tau function we can write

$$\psi_k = \sum_{i=1}^{N} \frac{w_{k,i}}{x - x_i}, \quad k \geq 1, \tag{6.9}$$

and in particular, for $k = 1$ formula (6.8) gives

$$w_{1,i} = -x_i'. \tag{6.10}$$

Putting (6.1) and (6.7) in (2.6), and comparing the coefficients of z^{-k}, we obtain

$$\psi_{k+1} + \frac{\partial \psi_k}{\partial t_1} = \psi_{k+1}(xq) + D_{q,x}\psi_k + x(q-1)\sum_{i=1}^{N} \frac{x_i'}{(x - x_i)(xq - x_i)}\psi_k. \tag{6.11}$$

For $k > 0$, the coefficient of $(x - x_i/q)^{-1}$ in (6.11) gives the reccurence relation

$$-\frac{w_{k+1,i}}{q} = \frac{1+x_i'}{x_i(q-1)} w_{k,i} - \sum_{j\neq i} \frac{x_i'}{x_i - qx_j} w_{k,j}, \tag{6.12}$$

and the coefficient of $(x - x_i)^{-1}$ in (6.11) gives

$$w_{k+1,i} + \frac{\partial w_{k,i}}{\partial t_1} = \left(-\frac{1+x_i'}{x_i(q-1)} + \sum_{j\neq i} \frac{(q-1)x_i x_j'}{(x_i - x_j)(x_i q - x_j)} \right) w_{k,i} + \sum_{j\neq i} \frac{x_i'}{x_i - x_j} w_{k,j}. \tag{6.13}$$

Denoting $X = \mathrm{diag}(x_1, x_2, \ldots, x_N)$, $w_k = (w_{k,1}, \ldots, w_{k,N})^{\mathrm{t}}$, $\mathbf{e} = (1, 1, \ldots, 1)^{\mathrm{t}}$ we have from (6.10) and (6.12)

$$w_k = (-qY)^{k-1} X' \mathbf{e}, \tag{6.14}$$

where Y is the q-deformed Calogero–Moser matrix defined by (6.4) and (6.5). Eliminating $w_{k+1,i}$ from Eqs. (6.12) and (6.13) we obtain

$$\frac{\partial w_{k,i}}{\partial t_1} = \left(\frac{1+x_i'}{x_i} + \sum_{j\neq i} \frac{(q-1)x_i x_j'}{(x_i - x_j)(x_i q - x_j)} \right) w_{k,i} - \sum_{j\neq i} \frac{(q-1)x_i x_i'}{(x_i - x_j)(x_i - qx_j)} w_{k,j}. \tag{6.15}$$

Let

$$\Psi^{q*}(x, t, z) = \left(1 + \sum_{k=1}^{\infty} \psi_k^* z^{-k} \right) \mathrm{e}_{1/q}^{-xz} \exp\left(-\sum_{k=1}^{\infty} t_k z^k \right)$$

be the q-adjoint wave function (see [25]). Writing ψ_k^* as

$$\psi_k^* = \sum_{i=1}^{N} \frac{w_{k,i}^*}{x - x_i}$$

and comparing the coefficients of $(x - qx_i)^{-1}$ in

$$\partial_1 \Psi^{q*} = -(L|_{x/q})_+^* \Psi^{q*} = (D_{1/q,x} - a_0(x/q))\Psi^{q*},$$

we obtain as above

$$w_k^* = -X'(-Y^t)^{k-1} \mathbf{e}, \tag{6.16}$$

where $w_k^* = (w_{k,1}^*, w_{k,2}^*, \ldots, w_{k,N}^*)^t$. Now, we compute the coefficients of $D_{q,x}^{-1}$ in the equation

$$\partial_n S = -(S D_{q,x}^n S^{-1})_- S. \tag{6.17}$$

The left-hand side is

$$\partial_n S = \sum_{k=1}^{\infty}\sum_{i=1}^{N}\left(\frac{(\partial_n x_i)w_{k,i}}{(x-x_i)^2}+\frac{\partial_n w_{k,i}}{x-x_i}\right)D_{q,x}^{-k}$$
$$=\left(\frac{x_i'(\partial_n x_i)}{(x-x_i)^2}+\frac{\partial_n x_i'}{x-x_i}\right)D_{q,x}^{-1}+\mathrm{O}(D_{q,x}^{-2}). \tag{6.18}$$

On the other hand, from the definition of the q-adjoint wave function we have

$$S^{-1}=\sum_{j=0}^{\infty}D_{q,x}^{-j}\cdot\psi_j^*(xq),$$

so the right-hand side of (6.17) becomes

$$-(SD_{q,x}^{n}S^{-1})_{-}S=-\left(\sum_{k+l\geq n+1}\psi_k D_{q,x}^{n-k-l}\cdot\psi_l^*(xq)\right)\left(1+\sum_{k\geq 1}\psi_k D_{q,x}^{-k}\right)$$
$$=-\sum_{k+l=n+1}\psi_k\psi_l^* D_{q,x}^{-1}+\mathrm{O}(D_{q,x}^{-2}),$$

and thus

$$\frac{x_i'\partial_n x_i}{(x-x_i)^2}+\frac{\partial_n x_i'}{x-x_i}=-\sum_{k+l=n+1}\psi_l^*\psi_k.$$

Comparing the coefficients of $(x-x_i)^{-2}$ in the above identity, and using (6.14) and (6.16), we get

$$x_i'(\partial_n x_i)=-\sum_{k=1}^{n}w_{n+1-k,i}^{*}w_{k,i}=\sum_{k=1}^{n}w_{n+1-k}^{*t}E_{ii}w_k$$
$$=(-1)^{n+1}\sum_{k=1}^{n}q^{k-1}\,\mathbf{e}^t Y^{n-k}X'E_{ii}Y^{k-1}X'\mathbf{e}.$$

Here, as usual, E_{ij} denotes the matrix with 1 at the $(i,\ j)$th entry, with all other entries zero. Since $X'E_{ii}=x_i'E_{ii}$, we can cancel x_i' and rewrite the above equality as

$$\partial_n x_i=(-1)^{n+1}\,\mathrm{tr}\left(\sum_{k=1}^{n}q^{k-1}Y^{n-k}E_{ii}Y^{k-1}X'\begin{pmatrix}1&\dots&1\\ \vdots&&\vdots\\ 1&\dots&1\end{pmatrix}\right).$$

Finally, replacing $X'\begin{pmatrix}1&\dots&1\\ \vdots&&\vdots\\ 1&\dots&1\end{pmatrix}$ by $-(E+XY-qYX)$, and using the elementary properties of the trace operator we get

$$\begin{aligned}\partial_n x_i &= (-1)^n \operatorname{tr}\left(\sum_{k=1}^{n} q^{k-1} Y^{n-k} E_{ii} Y^{k-1}(E + XY - qYX)\right)\\ &= (-1)^n \left(\sum_{k=1}^{n} q^{k-1} \operatorname{tr}(E_{ii}Y^{n-1}) + \sum_{k=1}^{n} q^{k-1} \operatorname{tr}(Y^{n+1-k}E_{ii}Y^{k-1}X)\right.\\ &\quad \left.-\sum_{k=1}^{n} q^{k} \operatorname{tr}(Y^{n-k}E_{ii}Y^{k}X)\right)\\ &= (-1)^n [n]_q \operatorname{tr}((E_{ii} + (1-q)E_{ii}XY)Y^{n-1}). \end{aligned} \tag{6.19}$$

But since

$$\frac{\partial Y}{\partial y_i} = E_{ii} + (1-q)E_{ii}XY,$$

(6.19) gives the first equation in (6.6). To get the second equation, we need to represent 'nicely' the first flow $\partial/\partial t_1$ on the matrix Y, which is the content of the next lemma.

Lemma 6.2. *The first flow $\partial/\partial t_1$ can be written in the Lax form*

$$\frac{\partial Y}{\partial t_1} = [Y, M], \tag{6.20}$$

where M is another deformation of the Calogero–Moser matrix given by

$$M_{ij} = -\frac{x_i'}{x_i - x_j} \quad \text{for } i \neq j, \qquad M_{ii} = \frac{1 + x_i'}{x_i(q-1)} + \sum_{k\neq i}\left(\frac{x_k'}{x_i q - x_k} - \frac{x_k'}{x_i - x_k}\right).$$

Proof of Lemma 6.2. The equality (6.20) can be checked directly, using (6.10) and (6.15) for $k = 1$, and the definition (6.4) and (6.5) of Y. □

We can now finish the proof of Theorem 6.1. From (6.3),(6.19) and Lemma 6.2 one can easily deduce

$$\frac{\partial y_i}{\partial t_n} = (-1)^n [n]_q \operatorname{tr}(BY^{n-1}), \tag{6.21}$$

where

$$\begin{aligned} B &= \frac{1}{(q-1)x_i x_i'}[E_{ii}, \hat{M}]\\ &\quad + \frac{1}{x_i}E_{ii}Y + \frac{1}{x_i'}(\hat{M}E_{ii}Y - YE_{ii}\hat{M}) + \frac{1}{(q-1)x_i}\sum_j \frac{\partial \log A_i}{\partial x_j}E_{jj}\\ &\quad - \left(\sum_j \frac{x_j}{x_i}\frac{\partial \log A_i}{\partial x_j}E_{jj}\right)Y - \frac{y_i}{x_i}(E_{ii} + (1-q)x_i E_{ii}Y), \end{aligned}$$

with $\hat{M}_{jk} = (1 - \delta_{jk})M_{jk}$. On the other hand, a direct computation shows that

$$B+\frac{\partial Y}{\partial x_i}=\left[\sum_{j\neq i}\frac{\partial\log A_j}{\partial x_i}E_{jj}+\frac{1}{(1-q)x_i}E_{ii},Y\right],\tag{6.22}$$

which combined with (6.21) gives the second equation in (6.6).

Remark 6.3 (The limiting case $q=1$). *We should note that our choice of 'dual' variables* $\{y_i\}$ *does not reduce exactly to the standard one* $\{\xi_i\}$ *with* $\xi_i=\partial x_i/\partial t_2$ *in the classical case* $q=1$. *Indeed, in the limit* $q\to 1$, *from* (6.4) and (6.5), *it follows that*

$$\lim_{q\to 1}Y_{ij}=\frac{1}{x_i-x_j}\quad\text{for } i\neq j,\qquad \lim_{q\to 1}Y_{ii}=\frac{\partial x_i}{\partial t_2}=y_i+\sum_{j\neq i}\frac{1}{x_j-x_i},$$

i.e.

$$\xi_i=y_i+\sum_{j\neq i}\frac{1}{x_j-x_i}.$$

From these relations, it is not difficult to see that the system of equations (6.6) *is equivalent to Eq.* (3) *in* [40].

Acknowledgements

I wish to thank Professor Luc Haine, for his collaboration in [24] which inspired the present work, as well as for the numerous helpful ideas and discussions. I would like to thank also the unknown referee for his valuable suggestions.

References

[1] M. Adler, E. Horozov, P. van Moerbeke, The solution to the q-KdV equation, Phys. Lett. A 242 (1998) 139–151.
[2] M. Adler, P. van Moerbeke, Birkoff strata, Bäcklund transformations, and regularization of isospectral operators, Adv. in Math. 108 (1994) 140–204.
[3] H. Airault, H.P. McKean, J.K. Moser, Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem, Commun. Pure Appl. Math. 30 (1977) 95–148.
[4] R. Askey, J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Am. Math. Soc. 319 (1985).
[5] B. Bakalov, E. Horozov, M. Yakimov, Bäcklund–Darboux transformations in Sato's Grassmannian, Serdica Math. J. 22 (1996) 571–588.
[6] B. Bakalov, E. Horozov, M. Yakimov, General methods for constructing bispectral operators, Phys. Lett. A 222 (1996) 59–66.
[7] B. Bakalov, E. Horozov, M. Yakimov, Bispectral algebras of commuting ordinary differential operators, Commun. Math. Phys. 190 (1997) 331–373.
[8] B. Bakalov, E. Horozov, M. Yakimov, Highest weight modules over $W_{1+\infty}$ algebra and the bispectral problem, Duke Math. J. 93 (1998) 41–72.
[9] Y.Y. Berest, A.P. Veselov, Huygens principle and integrability, Russian Math. Surveys 49 (1994) 5–77.
[10] S. Bochner, Über Sturm–Liouvillesche polynomsysteme, Math. Z. 29 (1929) 730–736.
[11] J.L. Burchnall, T.W. Chaundy, Commutative ordinary differential operators, Proc. London Math. Soc. 21 (1923) 420–440.

[12] J.L. Burchnall, T.W. Chaundy, Commutative ordinary differential operators, Proc. Roy. Soc. London Ser. A 118 (1928) 557–583.
[13] J.L. Burchnall, T.W. Chaundy, Commutative ordinary differential operators, Proc. Roy. Soc. London Ser. A 134 (1932) 471–485 .
[14] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations, in: M. Jimbo, T. Miwa (Eds.), Proceedings of the RIMS Symposium on Nonlinear Integrable Systems — Classical and Quantum Theory, Kyoto, 1981, World Scientific, Singapore, 1983, pp. 39–119.
[15] J.J. Duistermaat, F.A. Grünbaum, Differential equations in the spectral parameter, Commun. Math. Phys. 103 (2) (1986) 177–240.
[16] E. Frenkel, Deformations of the KdV hierarchy and related soliton equations, Int. Math. Res. Notices 2 (1996) 55–76.
[17] E. Frenkel, N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and $\mathcal{W}$-algebras, Commun. Math. Phys. 178 (1996) 237–264.
[18] F.A. Grünbaum, L. Haine, Orthogonal polynomials satisfying differential equations: the role of the Darboux transformation, in: D. Levi, L. Vinet, P. Winternitz (Eds.), Symmetries and Integrability of Difference Equations, Estérel, 1994, CRM Proceedings, Lecture Notes, Vol. 9, American Mathematical Society, Providence, RI, 1996, pp. 143–154.
[19] F.A. Grünbaum, L. Haine, A theorem of Bochner, revisited, in: I.M. Gelfand, A.S. Fokas (Eds.), Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman, Progr. Nonlinear Differential Equations, Vol. 26, Birkhäuser, Boston, 1996, pp. 143–172.
[20] F.A. Grünbaum, L. Haine, The q-version of a theorem of Bochner, J. Comput. Appl. Math. 68 (1996) 103–114.
[21] F.A. Grünbaum, L. Haine, Bispectral Darboux transformations: An extension of the Krall polynomials, Int. Math. Res. Notices 8 (1997) 359–392.
[22] F.A. Grünbaum, L. Haine, Some functions that generalize the Askey–Wilson polynomials, Commun. Math. Phys. 184 (1997) 173–202.
[23] L. Haine, Beyond the classical orthogonal polynomials, in: J. Harnad, A. Kasman (Eds.), The Bispectral Problem, Montréal, 1997, CRM Proceedings, Lecture Notes, Vol. 14, American Mathematical Society, Providence, RI, 1998, pp. 47–65.
[24] L. Haine, P. Iliev, The bispectral property of a q-deformation of the Schur polynomials and the q-KdV hierarchy, J. Phys. A 30 (1997) 7217–7227.
[25] P. Iliev, Tau function solutions to a q-deformation of the KP hierarchy, Lett. Math. Phys. 44 (1998) 187–200.
[26] P. Iliev, Algèbres commutatives d'opérateurs aux q-différences et systèmes de Calogero–Moser, C.R. Acad. Sci. Paris Série I t. 329 (1999) 877–882.
[27] A. Kasman, Bispectral KP solutions and linearization of Calogero–Moser particle systems, Commun. Math. Phys. 172 (1995) 427–448.
[28] A. Kasman, M. Rothstein, Bispectral Darboux transformations: the generalized Airy case, Physica D 102 (3/4) (1997) 159–176.
[29] D. Kazhdan, B. Kostant, S. Sternberg, Hamiltonian group actions and dynamical systems of Calogero type, Commun. Pure Appl. Math. 31 (1978) 481–507.
[30] B. Khesin, V. Lyubashenko, C. Roger, Extensions and contractions of the Lie algebra of q-pseudodifferential symbols on the circle, J. Funct. Anal. 143 (1997) 55–97.
[31] I.M. Krichever, Methods of algebraic geometry in the theory of non-linear equations, Russian Math. Surveys 32 (1977) 185–213.
[32] I.M. Krichever, Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of n particles on a line, Functional Anal. Appl. 12 (1978) 59–61.
[33] J.I. Liberati, Bispectral property, Darboux transformation and the Grassmannian Gr^{rat}, Lett. Math. Phys. 41 (1997) 321–332.
[34] D. Mumford, An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg–de Vries equation and related non-linear equations, in: M. Nagata (Ed.), Proceedings of Symposium on Algebraic Geometry, Kinokuniya, Tokyo, 1978.
[35] A.L. Pirozerski, M.A. Semenov-Tian-Shansky, Generalized q-deformed Gelfand–Dickey structures on the group of q-pseudodifference operators, Preprint QA/9811025.
[36] M. Reach, Difference equations for N-soliton solutions to KdV, Phys. Lett. A 129 (1988) 101–105.
[37] M. Sato, Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, Lect. Notes Numer. Appl. Anal. 5 (1982) 259–271.
[38] G. Segal, G. Wilson, Loop groups and equations of KdV type, Publ. Math. IHES 61 (1985) 5–65.

[39] T. Shiota, Characterization of Jacobian varieties in terms of soliton equations, Invent. Math. 83 (1986) 333–382.
[40] T. Shiota, Calogero–Moser hierarchy and KP hierarchy, J. Math. Phys. 35 (1994) 5844–5849.
[41] G. Wilson, Bispectral commutative ordinary differential operators, J. Reine Angew. Math. 442 (1993) 177–204.
[42] G. Wilson, Collisions of Calogero–Moser particles and an adelic Grassmannian (with an appendix by I.G. Macdonald), Invent. Math. 133 (1998) 1–41.
[43] J.P. Zubelli, On the polynomial τ-functions for the KP hierarchy and the bispectral property, Lett. Math. Phys. 24 (1992) 41–48.
[44] J.P. Zubelli, F. Magri, Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for the KdV, Commun. Math. Phys. 141 (1991) 329–351.